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# Quantum thetas on noncommutative $\mathbb{T}^d$ with general embeddings

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**Abstract**
In this paper, we construct quantum theta functions over noncommutative $\mathbb{T}^d$ with general embeddings. Manin has constructed quantum theta functions from the lattice embedding into vector space $\times$ finite group. We extend Manin's construction of quantum thetas to the case of general embedding of vector space $\times$ lattice $\times$ torus. It turns out that only for the vector space part of the embedding there exists the holomorphic theta vector, while for the lattice part there does not. Furthermore, the so-called quantum translations from embedding into the lattice part become non-additive, while those from the vector space part are additive.

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## 1. Introduction

In the quantization of the classical theta function, we encounter two types of objects. One is the theta vector introduced by Schwarz [1], which is a holomorphic element of a projective module over unitary quantum torus. The other is the quantum theta function introduced by Manin [2–5], which is an element of the function ring of quantum torus itself. This is a natural outcome if we consider the process of quantization, in which commutative physical observables become operators acting on the states. Namely, classically we have only one type of objects, observables, and then after quantization we come up with two types of objects, operators and states. This is exactly what happens here. In the classical case, a set of specific values of observables constitutes a state, and the classical theta function is just like a state function. On the other hand, the quantum theta functions and the theta vectors are the operators and state vectors, respectively, in the quantum case. Manin [4, 5] has shown that the Rieffel's algebra-valued inner(scalar) products [6] of theta vectors [7] obtained from the lattice embedding of the type $\mathbb{R}^p(\times F)$ for quantum torus satisfy the property of quantum theta function that he defined. Here, $d = 2p$ is the dimension of the relevant quantum torus and $F$ is a finite group. However, it was also shown in [6] that there is another type of lattice embedding for quantum torus, $\mathbb{R}^p \times \mathbb{Z}^q(\times F)$, where the dimension of the relevant quantum torus is $d = 2p + q$. Manin has left the construction of the quantum theta function for this case in question [5].

This type of nonzero $q$ embedding is intimately related to the Morita equivalence over noncommutative tori [8]. In [9], we investigated the symmetry of quantum torus, restricting ourselves to the symmetry of the algebra and its module, which is not related to the Morita equivalence. In that case, we only considered the embeddings with $q = 0$. However, to investigate the full symmetry of noncommutative tori including the Morita equivalence, we need to understand the behavior of modules from nonzero $q$ embeddings.

We have previously constructed the quantum theta function in the latter type of embeddings that Manin has left in question in the case of noncommutative $\mathbb{T}^4$ [10]. This paper is the extension of the work in [10] to higher dimensional tori, providing the general proof of the result of the $\mathbb{T}^4$ case extended to the arbitrary $\mathbb{T}^d$ case.

We first try to find the theta vector in the nonzero $q$ embedding, and end up with a conclusion that the holomorphic theta vector does not exist in a general sense. Then we try to construct the quantum theta function in this case. Because still there is a possibility that the Rieffel's scalar product with an element of non-holomorphic (partially holomorphic only for the $\mathbb{R}^p$-part) module in the second type of embedding satisfies the required property of the quantum theta function. Thus, we construct a quantum theta function via Rieffel's scalar product with an element of the module in the second type of embedding and find that it satisfies the requirement of quantum theta function.

The organization of the paper is as follows. In section 2, we construct the modules with general embeddings for quantum tori. In section 3, we construct the quantum theta functions evaluating the scalar products of the above obtained modules, and check the required conditions for the quantum theta function. In section 4, we conclude with discussion.

## 2. Lattice embedding of quantum torus

We first review the embedding of quantum torus [6] and a canonical construction of the module with an embedding of the type $\mathbb{R}^p$, of which the four-torus case was done explicitly in [11]. Then we proceed to the case with an embedding of the type $\mathbb{R}^p \times \mathbb{Z}^q$.

Recall that $\mathbb{T}^d_\theta$ is a deformed algebra of the algebra of smooth functions on the torus $\mathbb{T}^d$ with the deformation parameter $\theta$, which is a real $d \times d$ anti-symmetric matrix. This algebra is generated by operators $U_1, \ldots, U_d$ obeying the following relations:

$$U_j U_i = \mathrm{e}^{2\pi\mathrm{i}\theta_{ij}} U_i U_j \qquad \text{and} \qquad U_i^* U_i = U_i U_i^* = 1, \qquad i, j = 1, \ldots, d.$$

The above relations define the presentation of the involutive algebra

$$\mathcal{A}^d_\theta = \left\{ \sum a_{i_1=\ldots i_d} U_1^{i_1} \ldots U_d^{i_d} \,\middle|\, a = (a_{i_1\ldots i_d}) \in \mathcal{S}(\mathbb{Z}^d) \right\},$$

where $\mathcal{S}(\mathbb{Z}^d)$ is the Schwartz space of sequences with rapid decay.

Every projective module over a smooth algebra $\mathcal{A}^d_\theta$ can be represented by a direct sum of modules of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, the linear space of Schwartz functions on $\mathbb{R}^p \times \mathbb{Z}^q \times F$, where $2p+q = d$, and $F$ is a finite Abelian group. The module action is specified by operators on $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ and the commutation relation of these operators should be matched with that of elements in $\mathcal{A}^d_\theta$.

Recall that there is the dual action of the torus group $\mathbb{T}^d$ on $\mathcal{A}^d_\theta$ which gives a Lie group homomorphism of $\mathbb{T}^d$ into the group of automorphisms of $\mathcal{A}^d_\theta$. Its infinitesimal form generates a homomorphism of the Lie algebra $L$ of $\mathbb{T}^d$ into the Lie algebra of derivations of $\mathcal{A}_\theta^d$. Note that the Lie algebra $L$ is Abelian and is isomorphic to $\mathbb{R}^d$. Let $\delta : L \to \mathrm{Der}(\mathcal{A}_\theta^d)$ be the homomorphism. For each $X \in L$, $\delta(X) := \delta_X$ is a derivation, i.e., for $u, v \in \mathcal{A}_\theta^d$,

$$\delta_X(uv) = \delta_X(u)v + u\delta_X(v). \tag{1}$$

Derivations corresponding to the generators $\{e_1, \ldots, e_d\}$ of $L$ will be denoted by $\delta_1, \ldots, \delta_d$. For the generators $U_i$'s of $\mathbb{T}_\theta^d$, it has the following property:

$$\delta_i(U_j) = 2\pi \mathrm{i}\delta_{ij} U_j. \tag{2}$$

Let $D$ be a lattice in $\mathcal{G} = M \times \widehat{M}$, where $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$, and $\widehat{M}$ is its dual. Let $\Phi$ be an embedding map such that $D$ is the image of $\mathbb{Z}^d$ under the map $\Phi$. This determines a projective module to be denoted by $E$ [6]. If $E$ is a projective $\mathcal{A}_\theta^d$-module, a connection $\nabla$ on $E$ is a linear map from $E$ to $E \otimes L^*$ such that for all $X \in L$,

$$\nabla_X(\xi u) = (\nabla_X \xi)u + \xi\delta_X(u), \qquad \xi \in E, u \in \mathcal{A}_\theta^d. \tag{3}$$

It is easy to see that

$$[\nabla_i, U_j] = 2\pi \mathrm{i}\delta_{ij} U_j. \tag{4}$$

In the Heisenberg representation, the operators are defined by

$$\mathcal{U}_{(m,\hat{s})} f(r) = \mathrm{e}^{2\pi \mathrm{i}\langle r,\hat{s}\rangle} f(r+m) \tag{5}$$

for $(m, \hat{s}) \in D$, $r \in M$.

Now, we proceed to the construction of the module, first for the embedding with the type $M = \mathbb{R}^p$, then with the type $M = \mathbb{R}^p \times \mathbb{Z}^q$. Here we suppress the finite part for brevity. We consider the embeddings of canonical forms in the present section, and in the following section we will further consider the generalization of the result from the canonical embeddings.

For $M = \mathbb{R}^p$ with $2p = d$, we put the embedding map as follows via proper rearrangement of the basis,

$$\Phi_{\mathrm{irr}} = \begin{pmatrix} \Theta & 0 \\ 0 & I \end{pmatrix} := (x_{i,j}), \qquad \text{for} \quad i, j = 1, \ldots, d, \tag{6}$$

where $\Theta$ and $I$ belong to $\mathbb{R}^p$ and $\mathbb{R}^{p*}$, respectively, and are given by $p \times p$ diagonal matrices of the type

$$\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_p), \qquad I = (\delta_{ij}), \qquad i, j = 1, \ldots, p. \tag{7}$$

Then using expression (5) for the Heisenberg representation, we get

$$\begin{aligned} (U_j f)(s_1, \ldots, s_p) &:= (U_{e_j} f)(\vec{s}), \\ &\equiv \exp\left(2\pi \mathrm{i} \sum_{k=1}^{p} s_k x_{k+p,j} + \sum_{k=1}^{p} x_{k,j} x_{p+k,j}\right) f(\vec{s} + \vec{x}_j), \\ &\text{for} \quad j = 1, \ldots, 2p, \end{aligned} \tag{8}$$

where $\vec{s} = (s_1, \ldots, s_p)$, $\vec{x}_j = (x_{1,j}, \ldots, x_{p,j})$ and $\vec{s}, \vec{x}_j \in \mathbb{R}^p$.

This can be redisplayed as

$$\begin{aligned} &(U_j f)(\vec{s}) = f(\vec{s} + \vec{\theta}), \\ &(U_{j+p} f)(\vec{s}) = \mathrm{e}^{2\pi \mathrm{i} s_j} f(\vec{s}), \qquad \text{for} \quad j, k = 1, \ldots, p, \end{aligned} \tag{9}$$

where $\vec{\theta} = (\theta_1, \ldots, \theta_p)$. One can see that they satisfy

$$U_j U_{j+p} = \mathrm{e}^{2\pi \mathrm{i}\theta_j} U_j U_{j+p}, \tag{10}$$

and otherwise $U_j U_k = U_k U_j$.

For the embedding of the type $M = \mathbb{R}^p \times \mathbb{Z}^q$ where $2p + q = d$, we put the embedding map of the canonical form as follows:

$$\Phi_{\mathrm{irr}} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q \\ 0 & 0 & \Delta \end{pmatrix} := (x_{i,j}), \qquad i = 1, \ldots, 2p + 2q, \quad j = 1, \ldots, 2p + q, \tag{11}$$

where $\Theta$ and $I$ are the same as before that belong to $\mathbb{R}^p$ and $\mathbb{R}^{p*}$, respectively, and $Q$ and $\Delta$ are the $q \times q$ matrices that belong to $\mathbb{Z}^q$ and $T^q$, respectively. Then, the operators $U_j$ acting on the space $E := \mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ can be defined via Heisenberg representation (5), and we get

$$\begin{aligned}(U_j f)(s_1, \ldots, s_p, n_1, \ldots n_q) &:= (U_{e_j} f)(\vec{s}, \vec{n}), \\ &\equiv \mathrm{e}^{2\pi \mathrm{i}(\sum_{k=1}^{p} s_k x_{p+k,j} + \sum_{l=1}^{q} n_l x_{2p+q+l,j}) + \pi \mathrm{i}(\sum_{k=1}^{p} x_{k,j} x_{p+k,j} + \sum_{l=1}^{q} x_{2p+l,j} x_{2p+q+l,j})} f(\vec{s} + \vec{x}_{1j}, \vec{n} + \vec{x}_{2j}), \\ &\quad \text{for} \quad j = 1, \ldots, 2p + q,\end{aligned} \tag{12}$$

where $\vec{x}_{1j} = (x_{1,j}, \ldots, x_{p,j})$ and $\vec{x}_{2j} = (x_{2p+1,j}, \ldots, x_{2p+q,j})$ belonging to $\mathbb{R}^p, \mathbb{Z}^q$, respectively.

## 3. Quantum thetas

In this section, we first try to construct the theta vector by defining the connection with a complex structure for the embedding of the type $\mathbb{R}^p \times \mathbb{Z}^q$. Then, we construct the quantum theta function following the Manin's construction.

### *3.1. Theta vectors*

In the previous section, connections on a projective $\mathcal{A}_\theta^d$-module satisfies condition (4) and can be written as

$$U_j \nabla_i = \nabla_i U_j - 2\pi \mathrm{i} \delta_{ij} U_j, \qquad \text{for} \quad i, j = 1, \ldots, 2p + q. \tag{13}$$

**Proposition 1** (Rieffel)**.** *Relation (13) is satisfied with the connection $\nabla_j$ such that*

$$\begin{aligned}(\nabla_j f)(\vec{s}, \vec{n}) = &-2\pi \mathrm{i} \left( \sum_{k=1}^{p} B_{j,k} s_k f(\vec{s}, \vec{n}) + \sum_{l=1}^{q} B_{j,2p+l} n_l f(\vec{s}, \vec{n}) \right) \\ &+ \sum_{k=1}^{p} B_{j,p+k} \frac{\partial f}{\partial s_k}(\vec{s}, \vec{n}), \qquad \textit{for} \quad j = 1, \ldots, 2p + q,\end{aligned} \tag{14}$$

*where $\vec{s} = (s_1, \ldots, s_p)$, $\vec{n} = (n_1, \ldots, n_q)$, and the constants $B_{j,k} \in \mathbb{R}$ satisfy the following condition:*

$$\sum_{k=1}^{p} (B_{i,k} x_{k,j} + B_{i,p+k} x_{p+k,j}) + \sum_{l=1}^{q} B_{i,2p+l} x_{2p+l,j} = \delta_{ij}, \qquad i, j = 1, \ldots, 2p + q. \tag{15}$$

Condition (15) says that the matrix $B$ is the inverse matrix of $\tilde{X}$ where $\tilde{X}_{ij} = (x_{i,j})$ for $i, j = 1, \ldots, 2p + q$. Namely, the inverse matrix of the upper $(2p + q) \times (2p + q)$ part of the matrix $(x_{i,j})$ is the matrix $B$:

$$B = \tilde{X}^{-1} \qquad \text{and} \qquad \tilde{X} = \begin{pmatrix} \Theta & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q \end{pmatrix}, \tag{16}$$

where $\Theta, I, Q$ are given for the canonical form in (11).

We say that a noncommutative torus is equipped with a complex structure if the Lie algebra $L$ mentioned in section 2 is equipped with such a structure. A complex structure on $L$ can be considered as a decomposition of the complexification $L \oplus \mathrm{i}L$ of $L$ in a direct sum of two complex conjugate subspace $L^{1,0}$ and $L^{0,1}$. We denote by $\bar{\delta}_1, \ldots, \bar{\delta}_{d/2}$, a basis in $L^{0,1}$. One can express $\bar{\delta}_\alpha, \alpha = 1, \ldots, d/2$ in terms of $\delta_\beta, \beta = 1, \ldots, d$ which appeared in section 2 as $\bar{\delta}_\alpha = h_\alpha^\beta \delta_\beta$, where $h_\alpha^\beta$ is a complex $\frac{d}{2} \times d$ matrix. A complex structure on a $\mathcal{A}_\theta^d$-module $E$ can be defined as a collection of $\mathbb{C}$-linear operators $\overline{\nabla}_1, \ldots, \overline{\nabla}_{d/2}$ on $E$ satisfying

$$\overline{\nabla}_\alpha(af) = a\overline{\nabla}_\alpha(f) + \bar{\delta}_\alpha(a)f, a \in \mathcal{A}_\theta^d, f \in E. \tag{17}$$

A vector $f \in E$ is called holomorphic if

$$\overline{\nabla}_\alpha f = 0, \qquad \alpha = 1, \ldots, d/2. \tag{18}$$

Now, we assume that there exists a complex structure $T$ such that

$$\begin{pmatrix} \overline{\nabla}_1 \\ \vdots \\ \overline{\nabla}_{d/2} \end{pmatrix} = \big(T, I\big) \begin{pmatrix} \nabla_1 \\ \vdots \\ \nabla_d \end{pmatrix}, \tag{19}$$

where $T$ is a $\frac{d}{2} \times \frac{d}{2}$ complex matrix and $I$ is a $\frac{d}{2} \times \frac{d}{2}$ unit matrix. In the canonical embedding (11), the connection $\nabla_\beta$ is given by (14) and (16):

$$\begin{pmatrix} \nabla_1 \\ \vdots \\ \nabla_d \end{pmatrix} = \begin{pmatrix} \Theta^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} -2\pi \mathrm{i} s_1 \\ \vdots \\ -2\pi \mathrm{i} s_p \\ \frac{\partial}{\partial s_1} \\ \vdots \\ \frac{\partial}{\partial s_p} \\ -2\pi \mathrm{i} n_1 \\ \vdots \\ -2\pi \mathrm{i} n_q \end{pmatrix}. \tag{20}$$

If there exists a holomorphic vector $f(\vec{s}, \vec{n})$, then the following equation should be satisfied:

$$\begin{pmatrix} \overline{\nabla}_1 \\ \vdots \\ \overline{\nabla}_{d/2} \end{pmatrix} f = 0. \tag{21}$$

The above equation can be written as

$$(T, I) \begin{pmatrix} \Theta^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix} \begin{pmatrix} -2\pi \mathrm{i} s_1 \\ \vdots \\ -2\pi \mathrm{i} s_p \\ \frac{\partial}{\partial s_1} \\ \vdots \\ \frac{\partial}{\partial s_p} \\ -2\pi \mathrm{i} n_1 \\ \vdots \\ -2\pi \mathrm{i} n_q \end{pmatrix} f = 0. \tag{22}$$

To check the existence condition for the holomorphic vector, we let

$$(T,I)\begin{pmatrix}\Theta^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & Q^{-1}\end{pmatrix} := (A,C,F), \tag{23}$$

where $A$ and $C$ are the $\left(p+\frac{q}{2}\right)\times p$ matrices and $F$ is the $\left(p+\frac{q}{2}\right)\times q$ matrix. Then the required condition for $f$ is

$$2\pi\mathrm{i}\sum_{k=1}^{p}(A_{ik}s_k + F_{il}n_l)f = \sum_{k=1}^{p} C_{ik}\frac{\partial f}{\partial s_k}, \qquad \text{for}\quad i=1,\ldots,\frac{d}{2}=p+\frac{q}{2}. \tag{24}$$

The only possible function is of the form

$$f(\vec{s},\vec{n}) = \exp\left[2\pi\mathrm{i}\left(\frac{1}{2}\sum_{j,k=1}^{p} s_j\Omega_{jk}s_k + \sum_{k=1}^{p}\sum_{l=1}^{q} G_{lk}n_l s_k\right)\right], \tag{25}$$

where $\Omega^t = \Omega$. Then condition (24) becomes

$$\begin{aligned}\sum_{k=1}^{p} C_{ik}\Omega_{kj} &= A_{ij}, \qquad 1\leqslant i\leqslant p+\frac{q}{2},\quad 1\leqslant j\leqslant p,\\ \sum_{k=1}^{p} C_{ik}G_{lk} &= F_{il}, \qquad 1\leqslant i\leqslant p+\frac{q}{2},\quad 1\leqslant l\leqslant q.\end{aligned} \tag{26}$$

In other words,

$$C\Omega = A \qquad \text{and} \qquad CG^t = F. \tag{27}$$

Combining these two conditions and from (23), we obtain the following relation:

$$C(\Omega, I, G^t) = (A,C,F) = (T,I)\begin{pmatrix}\Theta^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & Q^{-1}\end{pmatrix}. \tag{28}$$

**Proposition 2.** *We consider the existence of the holomorphic vector in the canonical embeddings in three different cases.*

*(i) For $p\neq 0, q=0$, there is the unique holomorphic vector with $\Omega = T\Theta^{-1}$ which is symmetric and whose imaginary part is positive definite.*
*(ii) For $p\neq 0, q\neq 0$, the holomorphic vector does not exist.*
*(iii) For $p=0, q\neq 0$, the only possible one is the delta function at the origin.*

**Proof.** In the case (i), the consistency relation (28) is reduced to

$$C(\Omega, I) = (A,C) = (T,I)\begin{pmatrix}\Theta^{-1} & 0\\ 0 & I\end{pmatrix} = (T\Theta^{-1}, I). \tag{29}$$

Thus one can see immediately that $C = I$ and $\Omega = T\Theta^{-1}$. Since $\Omega$ is symmetric by construction, so is $T\Theta^{-1}$, and this is the necessary condition for the existence of holomorphic theta vector. Here, in order $f$ to be a Schwartz function, the imaginary part of $T\Theta^{-1}$ should be positive.

In the case (ii), the consistency relation (28) is

$$C(\Omega, I, G^t) = (T,I)\begin{pmatrix}\Theta^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & Q^{-1}\end{pmatrix}. \tag{30}$$

The above relation can be understood as linear maps from $\mathbb{C}^{2p+q} \to \mathbb{C}^{p} \to \mathbb{C}^{p+\frac{q}{2}}$ for the left and from $\mathbb{C}^{2p+q} \to \mathbb{C}^{2p+q} \to \mathbb{C}^{p+\frac{q}{2}}$ for the right. The right linear map is surjective since both $(T, I)$ and

$$\begin{pmatrix} \Theta^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}$$

are of full rank, while the left linear map cannot be surjective since it is maximally of rank $p$ which is strictly smaller than $p + \frac{q}{2}$.

In the case (iii), the consistency relation (28) becomes

$$(T, I)(Q^{-1}) \begin{pmatrix} -2\pi \mathrm{i} n_1 \\ \vdots \\ -2\pi \mathrm{i} n_q \end{pmatrix} f = 0. \tag{31}$$

If one can let $(T, I)(Q^{-1}) = F$ as defined in (23), where $T$ and $I$ are the $\frac{q}{2} \times \frac{q}{2}$ matrices and $Q$ is the $q \times q$ matrix, then the above condition can be written as

$$\left( \sum_{l=1}^{q} F_{il} n_l \right) f(\vec{n}) = 0, \qquad \text{for} \quad i = 1, \ldots, \frac{q}{2}. \tag{32}$$

If $f$ is a nontrivial solution, then $\sum_{l=1}^{q} F_{il} n_l = 0$ for all $i = 1, \ldots, \frac{q}{2}$. Since $F_{il} \in \mathbb{C}$, $(0, \ldots, 0)$ is the only solution for $\vec{n}$. Namely, $f$ can be nonzero only for $\vec{n} = (0, \ldots, 0)$, i.e., $f$ is a delta function at the origin. And (32), which is a re-phrasal of (21), tells us that $f$ has a non-vanishing solution only when the connection vanishes. In effect, one can say that the holomorphic vector does not exist in this case, either.

Now we consider the changes of the above result in the general set-up. First, consider the construction of the module from embeddings of the type $M = \mathbb{R}^p \times \mathbb{Z}^q$ where $2p + q = d$. Here again, we suppress the finite part for brevity. Let the embedding map be

$$\Phi := (x_{i,j}), \qquad i = 1, \ldots, 2p + 2q, \qquad j = 1, \ldots, 2p + q. \tag{33}$$

The operators $U_j$ acting on the space $E := \mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ can be defined via Heisenberg representation, and are given by equation (12) for more general values of $x_{i,j}$ given by the above embedding.

For the theta vectors, equation (15) tells us that the matrix $B$ is the inverse matrix of $\tilde{X}$ where $\tilde{X}_{ij} = (x_{i,j})$ for $i, j = 1, \ldots, 2p + q$. Namely, the matrix $\tilde{X}$ is the upper $(2p + q) \times (2p + q)$ square part of the matrix $\Phi$ and $B$ is its inverse matrix:

$$B = \tilde{X}^{-1}. \tag{34}$$

For a general complex structure, equation (19) can be written as

$$\begin{pmatrix} \overline{\nabla}_1 \\ \vdots \\ \overline{\nabla}_{d/2} \end{pmatrix} = (T_1, T_2) \begin{pmatrix} \nabla_1 \\ \vdots \\ \nabla_d \end{pmatrix}, \tag{35}$$

where $T_1$ and $T_2$ are $\frac{d}{2} \times \frac{d}{2}$ complex matrices with $d$ given by $2p + q$. And the connection $\nabla$ defined in (3) is given by (14):

$$\begin{pmatrix} \nabla_1 \\ \vdots \\ \nabla_d \end{pmatrix} = (B) \begin{pmatrix} -2\pi \mathrm{i} s_1 \\ \vdots \\ -2\pi \mathrm{i} s_p \\ \frac{\partial}{\partial s_1} \\ \vdots \\ \frac{\partial}{\partial s_p} \\ -2\pi \mathrm{i} n_1 \\ \vdots \\ -2\pi \mathrm{i} n_q \end{pmatrix}, \tag{36}$$

where $B$ is a $(2p+q) \times (2p+q)$ matrix defined by (34). Now, the condition for holomorphic vector (21) becomes

$$\left(T_1, T_2\right)(B) \begin{pmatrix} -2\pi \mathrm{i} s_1 \\ \vdots \\ -2\pi \mathrm{i} s_p \\ \frac{\partial}{\partial s_1} \\ \vdots \\ \frac{\partial}{\partial s_p} \\ -2\pi \mathrm{i} n_1 \\ \vdots \\ -2\pi \mathrm{i} n_q \end{pmatrix} f = 0. \tag{37}$$

To check the existence condition for the holomorphic vector, we let

$$(T_1, T_2)(B) := (A, C, F), \tag{38}$$

where $A$ and $C$ are the $\left(p + \frac{q}{2}\right) \times p$ matrices and $F$ is the $\left(p + \frac{q}{2}\right) \times q$ matrix. Then the holomorphic condition for $f$ given by (25) is the same as in (27), and in the above notation, we can write the following relation:

$$C(\Omega, I, G^t) = (A, C, F) = (T_1, T_2)(B). \tag{39}$$

$\square$

**Theorem 3.** *The existence of holomorphic vectors in the general embeddings is as follows:*

*(i) For $p \neq 0, q = 0$, the unique solution is given by*

$$\Omega = (T_1 B_{12} + T_2 B_{22})^{-1} (T_1 B_{11} + T_2 B_{21}),$$

*where*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \qquad B_{i,j} \text{ is the } p \times p \text{ matrix,} \tag{40}$$

*with the following three conditions: (1) there should exist an inverse of the matrix $(T_1 B_{12} + T_2 B_{22})$, (2) the matrix $(T_1 B_{12} + T_2 B_{22})^{-1} (T_1 B_{11} + T_2 B_{21})$ should be symmetric and (3) $\mathrm{Im}((T_1 B_{12} + T_2 B_{22})^{-1} (T_1 B_{11} + T_2 B_{21})) > 0$.*

*(ii) For $p \neq 0, q \neq 0$, there does not exist holomorphic vector.*
*(iii) For $p = 0, q \neq 0$, the only possible solution is the delta function at the origin.*

**Proof.** In the case (i), the consistency relation (39) is reduced to

$$C(\Omega, I) = (A, C) = (T_1, T_2)\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = (T_1B_{11} + T_2B_{21}, T_1B_{12} + T_2B_{22}), \tag{41}$$

where we write the matrix $B$ in $2 \times 2$ block form with each block being a $p \times p$ matrix. Here, $\Omega$ is given by

$$\Omega = (T_1B_{12} + T_2B_{22})^{-1}(T_1B_{11} + T_2B_{21}).$$

In order to have a holomorphic theta vector the following conditions should be satisfied: (1) there should exist an inverse of the matrix $(T_1B_{12} + T_2B_{22})$, (2) the matrix $(T_1B_{12} + T_2B_{22})^{-1}(T_1B_{11} + T_2B_{21})$ should be symmetric, since $\Omega$ is symmetric by construction, and (3) $\mathrm{Im}((T_1B_{12} + T_2B_{22})^{-1}(T_1B_{11} + T_2B_{21})) > 0$ in order $f$ to be a Schwartz function.

In the case (ii), the consistency relation (30) becomes

$$C(\Omega, I, G^t) = (T_1, T_2)B. \tag{42}$$

The above relation can be understood as before in terms of linear maps from $\mathbb{C}^{2p+q} \to \mathbb{C}^p \to \mathbb{C}^{p+\frac{q}{2}}$ for the left, and from $\mathbb{C}^{2p+q} \to \mathbb{C}^{2p+q} \to \mathbb{C}^{p+\frac{q}{2}}$ for the right. The right linear map is surjective since both $(T_1, T_2)$ and $B$ are of full rank, while the left linear map cannot be surjective since it is maximally of rank $p$ which is strictly smaller than $p + \frac{q}{2}$ as before.

In the case (iii), relation (31) becomes

$$(T_1, T_2)(B)\begin{pmatrix} -2\pi i n_1 \\ \vdots \\ -2\pi i n_q \end{pmatrix} f = 0. \tag{43}$$

If one can let $(T_1, T_2)B = F$ as defined in (38), where $T_1$ and $T_2$ are the $\frac{q}{2} \times \frac{q}{2}$ matrices and $B$ is the $q \times q$ matrix, then the above condition can be written as

$$\left(\sum_{l=1}^{q} F_{il}n_l\right) f(\vec{n}) = 0, \qquad \text{for} \quad i = 1, \ldots, \frac{q}{2}. \tag{44}$$

In the same vein, should $f$ be a nontrivial solution, then $\sum_{l=1}^{q} F_{il}n_l = 0$ for all $i = 1, \ldots, \frac{q}{2}$ as before. Thus, $f$ can be nonzero only for $\vec{n} = (0, \ldots, 0)$, and (44), a re-phrasal of (21), tells us that $f$ can be a non-vanishing solution only when the connection vanishes. Therefore, the holomorphic vector does not exist in this case. $\square$

The above analysis shows that one cannot have a holomorphic vector over totally complexified $\mathbb{T}^d_\theta$ in the embedding of $M = \mathbb{R}^p \times \mathbb{Z}^q$ with nonzero $p$ and $q$. This can be remedied by giving a complex structure only over the continuous part of the embedding space, i.e., by giving a complex structure to the connection components over $\mathbb{R}^p \times \mathbb{R}^{p*}$. Now, we implement this as follows:

$$\begin{aligned} \begin{pmatrix} \overline{\nabla}_1 \\ \vdots \\ \overline{\nabla}_p \end{pmatrix} &= (T_1, T_2)\begin{pmatrix} \nabla_1 \\ \vdots \\ \nabla_{2p} \end{pmatrix}, \\ \overline{\nabla}_{p+1} &= \nabla_{2p+1}, \\ &\vdots \\ \overline{\nabla}_{p+q} &= \nabla_{2p+q}, \end{aligned} \tag{45}$$

where $T_1$ and $T_2$ are the $p \times p$ complex matrices and give the complex structure over $\mathbb{R}^p \times \mathbb{R}^{p*}$. Then, the holomorphic vector over this part satisfies

$$\begin{pmatrix} \overline{\nabla}_1 \\ \vdots \\ \overline{\nabla}_p \end{pmatrix} f(\vec{s}, \vec{n}) = 0, \tag{46}$$

whose solution is given by

$$f(\vec{s}, \vec{n}) = \exp\left(\pi \mathrm{i} \sum_{j,k=1}^{p} s_j \Omega_{jk} s_k\right) g(\vec{n}).$$

Since $f$ belongs to $\mathcal{S}(\mathbb{R}^p) \otimes \mathcal{S}(\mathbb{Z}^q)$, $g(\vec{n})$ belongs to $\mathcal{S}(\mathbb{Z}^q)$ and has to be a Schwartz function. Here, we choose a simple Schwartz function for $g(\vec{n})$, and write the function $f(\vec{s}, \vec{n})$ as

$$f(\vec{s}, \vec{n}) = \exp\left[\pi \mathrm{i} \sum_{j,k=1}^{p} s_j \Omega_{jk} s_k - \frac{\pi}{2} \sum_{i=1}^{\frac{q}{2}} \left(n_i^2 + n_{\frac{q}{2}+i}^2\right)\right], \tag{47}$$

where $\operatorname{Im} \Omega > 0$.

### 3.2. *Quantum theta functions*

Before considering the quantum theta function, we first review the algebra-valued inner product on a bimodule after Rieffel [6]. Let $M$ be any locally compact Abelian group, and $\widehat{M}$ be its dual group, and let $\mathcal{G} \equiv M \times \widehat{M}$. Let $\pi$ be a representation of $\mathcal{G}$ on $L^2(M)$ such that

$$\pi_x \pi_y = \alpha(x, y) \pi_{x+y} = \alpha(x, y) \overline{\alpha}(y, x) \pi_y \pi_x \qquad \text{for} \quad x, y \in \mathcal{G}, \tag{48}$$

where $\alpha$ is a map $\alpha : \mathcal{G} \times \mathcal{G} \to \mathbb{C}^*$ satisfying

$$\alpha(x, y) = \alpha(y, x)^{-1}, \qquad \alpha(x_1 + x_2, y) = \alpha(x_1, y)\alpha(x_2, y),$$

and $\overline{\alpha}$ denotes the complex conjugation of $\alpha$. Let $D$ be a discrete subgroup of $\mathcal{G}$. We define $\mathcal{S}(D)$ as the space of Schwartz functions on $D$. For $\Psi \in \mathcal{S}(D)$, it can be expressed as $\Psi = \sum_{w \in D} \Psi(w) e_{D,\alpha}(w)$ where $e_{D,\alpha}(w)$ is a delta function with support at $w$ and obeys the following relation:

$$e_{D,\alpha}(w_1) e_{D,\alpha}(w_2) = \alpha(w_1, w_2) e_{D,\alpha}(w_1 + w_2). \tag{49}$$

For Schwartz functions $f, g \in \mathcal{S}(M)$, the algebra $(\mathcal{S}(D))$ valued inner product is defined as

$${}_D\langle f, g\rangle \equiv \sum_{w \in D} {}_D\langle f, g\rangle(w) e_{D,\alpha}(w), \tag{50}$$

where

$${}_D\langle f, g\rangle(w) = \langle f, \pi_w g\rangle.$$

Here, the scalar product of the type $\langle f, p\rangle$ above with $f, p \in L^2(M)$ denotes the following:

$$\langle f, p\rangle = \int f(x_1) \overline{p(x_1)} \, \mathrm{d}\mu_{x_1} \qquad \text{for} \quad x = (x_1, x_2) \in M \times \widehat{M}, \tag{51}$$

where $\mu_{x_1}$ represents the Haar measure on $M$ and $\overline{p(x_1)}$ denotes the complex conjugation of $p(x_1)$. The $\mathcal{S}(D)$-valued inner product can be represented as

$${}_D\langle f, g\rangle = \sum_{w \in D} \langle f, \pi_w g\rangle e_{D,\alpha}(w). \tag{52}$$

For $\Psi \in \mathcal{S}(D)$ and $f \in \mathcal{S}(M)$, then $\pi(\Psi) f \in \mathcal{S}(M)$ can be written as [6]

$$(\pi(\Psi) f)(m) = \sum_{w \in D} \Psi(w)(\pi_w f)(m), \tag{53}$$

where $m \in M$, $w \in D \subset M \times \widehat{M}$.

Now, we consider the Manin's quantum theta function $\Theta_D$ [3–5] for the embedding into vector space. In [5], the quantum theta function was defined via algebra-valued inner product up to a constant factor [12],

$${}_D\langle f, f\rangle \sim \Theta_D, \tag{54}$$

where $f$ used in the Manin's construction [5] was a simple Gaussian theta vector

$$f = \mathrm{e}^{\pi \mathrm{i} x_1^t T x_1}, \qquad x_1 \in M. \tag{55}$$

Here, $T$ is a complex structure given by a complex skew-symmetric matrix. With a given complex structure $T$, a complex variable $\underline{x} \in \mathbb{C}^p$ can be introduced via

$$\underline{x} \equiv T x_1 + x_2, \tag{56}$$

where $x = (x_1, x_2) \in M \times \widehat{M}$.

Based on the defining concept for the quantum theta function (54), one can define the quantum theta function $\Theta_D$ in the noncommutative $\mathbb{T}^{2p}$ case as

$${}_D\langle f, f\rangle = \frac{1}{\sqrt{2^p \det(\operatorname{Im} T)}} \Theta_D, \tag{57}$$

where $f$ is given by (55) and $T$ corresponds to $\Omega$ in (47). According to (50), the $\mathcal{S}(D)$-valued inner product (57) can be written as

$${}_D\langle f, f\rangle = \sum_{h \in D} \langle f, \pi_h f\rangle e_{D,\alpha}(h). \tag{58}$$

In [5], Manin showed that the quantum theta function defined in (57) is given by

$$\Theta_D = \sum_{h \in D} \mathrm{e}^{-\frac{\pi}{2} H(\underline{h}, \underline{h})} e_{D,\alpha}(h), \tag{59}$$

where

$$H(\underline{g}, \underline{h}) \equiv \underline{g}^t (\operatorname{Im} T)^{-1} \underline{h}^*$$

with $\underline{h}^* = \overline{T} h_1 + h_2$ denoting the complex conjugate of $\underline{h}$. At the same time, it also satisfies a quantum version of the translation action for classical theta functions [3]:

$$^\forall g \in D,\ C_g e_{D,\alpha}(g) x_g^*(\Theta_D) = \Theta_D, \tag{60}$$

where $C_g$ is defined by

$$C_g = \mathrm{e}^{-\frac{\pi}{2} H(\underline{g}, \underline{g})}$$

and the action of $x_g^*$, 'quantum translation', is given by

$$x_g^*(e_{D,\alpha}(h)) = \mathrm{e}^{-\pi H(\underline{g}, \underline{h})} e_{D,\alpha}(h). \tag{61}$$

In [3], Manin has also required that the factor $C_g$, $g \in D$ appearing in the quantum translation $x_g^*$ has to satisfy the following relation under a combination of quantum translations for consistency:

$$\frac{C_{g+h}}{C_g C_h} = \mathcal{T}_g(h) \alpha(g, h). \tag{62}$$

Here, $\alpha(g,h)$ is the cocycle appearing in (49) and $\mathcal{T}_g(h)$ is a generalized expression of the factor that appears by quantum translation:

$$x_g^*(e_{D,\alpha}(h)) \equiv \mathcal{T}_g(h) e_{D,\alpha}(h). \tag{63}$$

The proof of the functional relation (60) in this embedding case with quantum translation (61) was shown in [5].

We now construct the quantum theta function for the general embedding of $\mathbb{R}^p \times \mathbb{Z}^q$ for $2p+q=d$, using the function obtained in the previous section. With the function $f(\vec{s},\vec{n})$ given by (47), we evaluate the quantum theta function *a la* Manin:

$$\frac{1}{\sqrt{2^p \det(\operatorname{Im}\Omega)}}\hat{\Theta}_D = {}_D\langle f, f\rangle, \tag{64}$$

where $\Omega$ is a 'complex structure' over the continuous part of the embedding space, as is determined in the previous section including the noncommutativity parameters. We will see that the quantum theta function obtained in this way also satisfies the Manin-type functional relation with the modified quantum translation

$$^\forall g \in D, \qquad \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) = \hat{\Theta}_D, \tag{65}$$

where $\hat{C}_g$, $\hat{x}_g^*$ are to be defined below.

To evaluate the quantum theta function (64), we calculate the scalar product inside the summation in (58) first. For that we first write the action of the operator $\pi_h$ on $f$ omitting the arrow which denotes a vector for brevity:

$$\pi_h f(s,n) = \mathrm{e}^{2\pi\mathrm{i}(w_{h2}\cdot s + r\cdot n) + \pi\mathrm{i}(w_{h1}\cdot w_{h2} + m\cdot r)} f(s + w_{h1}, n+m), \tag{66}$$

where $h \in D$ is given by

$$h = (w_{h1}, w_{h2}, m, r) \in \mathbb{R}^p \times \mathbb{R}^{p*} \times \mathbb{Z}^q \times \mathbb{T}^q.$$

Then,

$$\begin{aligned}
\langle f, \pi_h f\rangle &= \sum_{n\in\mathbb{Z}^q}\int_{\mathbb{R}^p} \mathrm{d}s\, \mathrm{e}^{\pi[\mathrm{i}s^t\Omega s - \frac{1}{2}\sum_{i=1}^{\frac{q}{2}}(n_i^2 + n_{\frac{q}{2}+i}^2)]}\, \mathrm{e}^{\pi[-2\mathrm{i}(w_{h2}\cdot s + r\cdot n) - \mathrm{i}(w_{h1}\cdot w_{h2} + m\cdot r)]} \\
&\quad \times \mathrm{e}^{\pi[-\mathrm{i}(s+w_{h1})^t\Omega(s+w_{h1}) - \frac{1}{2}\sum_{i=1}^{\frac{q}{2}}((n_i+m_i)^2 + (n_{\frac{q}{2}+i} + m_{\frac{q}{2}+i})^2)]} \\
&= \int_{\mathbb{R}^p}\mathrm{d}s\,\mathrm{e}^{-2\pi[s^t(\operatorname{Im}\Omega)s + \mathrm{i}w_{h1}^t\overline{\Omega}s + \mathrm{i}w_{h2}\cdot s] - \mathrm{i}\pi[w_{h1}^t\overline{\Omega}w_{h1} + w_{h1}\cdot w_{h2}]} \\
&\quad \times \mathrm{e}^{-\frac{\pi}{2}\sum_{i=1}^{\frac{q}{2}}(m_i^2 + m_{\frac{q}{2}+i}^2) - \pi\mathrm{i}m\cdot r} \sum_{n\in\mathbb{Z}^q} \mathrm{e}^{-\frac{\pi}{2}\sum_{i=1}^{\frac{q}{2}}(n_i^2 + n_{\frac{q}{2}+i}^2) + 2\pi\mathrm{i}[n\cdot(-r + \frac{\mathrm{i}m}{2})]} \\
&= \prod_{j=1}^{q} b_{r_j,m_j}\int_{\mathbb{R}^p}\mathrm{d}s\,\mathrm{e}^{-2\pi[s^t(\operatorname{Im}\Omega)s + \mathrm{i}w_{h1}^t\overline{\Omega}s + \mathrm{i}w_{h2}\cdot s] - \mathrm{i}\pi[w_{h1}^t\overline{\Omega}w_{h1} + w_{h1}\cdot w_{h2}]},
\end{aligned} \tag{67}$$

where

$$b_{r_j,m_j} = \mathrm{e}^{-\frac{\pi}{2}m_j^2 - \pi\mathrm{i}m_j r_j}\theta\left(\tau = \mathrm{i}, z = -r_j + \frac{\mathrm{i}m_j}{2}\right), \qquad j = 1,\ldots,q. \tag{68}$$

Here, $\theta(\tau, z)$ is the classical theta function defined by

$$\theta(\tau,z) = \sum_{n\in\mathbb{Z}} \mathrm{e}^{\pi\mathrm{i}\tau n^2 + 2\pi\mathrm{i}nz}, \qquad \text{for} \quad \tau, z \in \mathbb{C}.$$

The integral in (67) is the same as that appeared in [5] and is given by

$$\frac{1}{\sqrt{2^p\det(\operatorname{Im}\Omega)}}\,\mathrm{e}^{-\frac{\pi}{2}H(\underline{w_h},\underline{w_h})}. \tag{69}$$

Thus we obtain the following result.

**Proposition 4.** *The quantum theta function $\hat{\Theta}_D$ obtained from $f$ in (47) is given by*

$$\hat{\Theta}_D = \sum_{h\in D} \widetilde{b}_h \, \mathrm{e}^{-\frac{\pi}{2}H(\underline{w_h},\underline{w_h})} e_{D,\alpha}(h), \tag{70}$$

*where*

$$\widetilde{b}_h = \prod_{j=1}^{q} b_{r_j,m_j} \tag{71}$$

*with $b_{r_j,m_j}$ given in (68).*

The above quantum theta function satisfy the Manin's functional relation under 'modified quantum translation' (65), and we get the following theorem.

**Theorem 5**

$$\forall g \in D, \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) = \hat{\Theta}_D,$$

*and the consistency condition (62) for $\hat{C}_g$. The above relation is satisfied if we assign*

$$\hat{C}_g = \widetilde{b}_g \, \mathrm{e}^{-\frac{\pi}{2}H(\underline{w_g},\underline{w_g})}, \tag{72}$$

*and $\hat{x}_g^*$ is defined by*

$$\hat{x}_g^*(e_{D,\alpha}(h)) = \hat{\mathcal{T}}_g(h) e_{D,\alpha}(h) \tag{73}$$

*with*

$$\hat{\mathcal{T}}_g(h) = \frac{\hat{C}_{g+h}}{\hat{C}_g \hat{C}_h \alpha(g,h)}. \tag{74}$$

**Proof.** Now, it is easy to show relation (65):

$$\begin{aligned}
\hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*(\hat{\Theta}_D) &= \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*\left(\sum_{h\in D} \widetilde{b}_h \, \mathrm{e}^{-\frac{\pi}{2}H(\underline{w_h},\underline{w_h})} e_{D,\alpha}(h)\right) \\
&= \hat{C}_g e_{D,\alpha}(g)\hat{x}_g^*\left(\sum_{h\in D} \hat{C}_h e_{D,\alpha}(h)\right) \\
&= \sum_{h\in D} \hat{C}_g \hat{C}_h e_{D,\alpha}(g)\hat{\mathcal{T}}_g(h) e_{D,\alpha}(h) \\
&= \sum_{h\in D} \hat{C}_{g+h} e_{D,\alpha}(g+h) = \hat{\Theta}_D,
\end{aligned}$$

where we used relation (72) in the second step, and relation (74) together with the cocycle condition (49) in the last step. □

**Remark.** Here we note that the quantum translations are not additive in this case:

$$\hat{x}_{g_1}^* \cdot \hat{x}_{g_2}^*(e_{D,\alpha}(h)) \neq \hat{x}_{g_1+g_2}^*(e_{D,\alpha}(h)). \tag{75}$$

On the other hand, the quantum translations in the Manin's case ($x_g^*$) (61) are additive:

$$x_{g_1}^* \cdot x_{g_2}^*(e_{D,\alpha}(h)) = x_{g_1+g_2}^*(e_{D,\alpha}(h)). \tag{76}$$

## 4. Conclusion

In this paper, we study the theta vector and the corresponding quantum theta function for noncommutative tori with general embeddings.

While the theta vector exists in the embedding into the vector space case ($\mathbb{R}^p$ type), there does not exist fully a holomorphic theta vector in the embedding into the lattice case ($\mathbb{Z}^q$ type). We construct a module which consists of holomorphic vectors for the vector space part and a plain Schwartz function for the lattice part in the case of mixed embedding ($\mathbb{R}^p \times \mathbb{Z}^q$ type). Manin has constructed the quantum theta functions only with holomorphic modules with embedding into vector space. And, it was not clear whether the partially holomorphic modules such as ours for mixed embeddings would yield the quantum theta functions that satisfy the Manin's requirement. The answer turns out to be yes.

There is one difference between the two types of quantum theta functions, Manin's and ours. In the Manin's quantum theta function, two consecutive 'quantum translations' are additive, while those of ours are not. This non-additivity is allowed by the consistency condition for the cocycle and quantum translation, (74).

In conclusion, we have shown that the quantum theta functions on noncommutative tori that satisfy the Manin's requirement can be constructed with any choice of the following embeddings: (1) into vector space times lattice, (2) into vector space and (3) into lattice. Our result for the cases (1) and (3) can be directly extended to the embeddings that include finite groups as was done in the Manin's work [5] for the case (2).

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